

$$\text{Calculer } S_n = \sum_{z \in \mathbb{U}_n} |z - 1|$$

Soit $z \in \mathbb{U}_n$, alors $z = e^{\frac{2ik\pi}{n}}$, $k \in \llbracket 0, n-1 \rrbracket$

$$\begin{aligned} |z - 1| &= |e^{\frac{2ik\pi}{n}} - 1| &&, k \in \llbracket 0, n-1 \rrbracket \\ &= |e^{\frac{ik\pi}{n}} (e^{\frac{ik\pi}{n}} - e^{-\frac{ik\pi}{n}})| &&, k \in \llbracket 0, n-1 \rrbracket \\ &= |2 \sin(\frac{k\pi}{n})| &&, k \in \llbracket 0, n-1 \rrbracket \end{aligned}$$

Étant sur la partie supérieure du cercle on a $\sin(\frac{k\pi}{n}) \geq 0$ pour $k \in \llbracket 0, n-1 \rrbracket$

$$\text{Donc } S_n = \sum_{z \in \mathbb{U}_n} |z - 1| = 2 \sum_{k=0}^{n-1} \sin(\frac{k\pi}{n})$$

$$\text{On pose } A_n = \sum_{k=0}^{n-1} \cos(\frac{k\pi}{n}) \text{ et } B_n = \sum_{k=0}^{n-1} \sin(\frac{k\pi}{n})$$

$$\begin{aligned} \text{On a } A_n + iB_n &= \sum_{k=0}^{n-1} \cos(\frac{k\pi}{n}) + i \sin(\frac{k\pi}{n}) = \sum_{k=0}^{n-1} \left(e^{\frac{i\pi}{n}} \right)^k \\ &= \frac{1 - \left(e^{\frac{i\pi}{n}} \right)^n}{1 - e^{\frac{i\pi}{n}}} \\ &= \frac{2}{e^{\frac{i\pi}{2n}} (e^{-\frac{i\pi}{2n}} - e^{\frac{i\pi}{2n}})} \end{aligned}$$

$$= \frac{2 \sin(\frac{\pi}{2n}) + 2i \cos(\frac{\pi}{2n})}{2 \sin(\frac{\pi}{2n})}$$

$$= 1 + i \frac{\cos(\frac{\pi}{2n})}{\sin(\frac{\pi}{2n})}$$

Par unicité de la partie Imaginaire on a donc:

$$S_n = 2 \operatorname{Im}(A_n + iB_n) = 2 \frac{\cos(\frac{\pi}{2n})}{\sin(\frac{\pi}{2n})}$$

Ce qui conclut l'exercice.